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Residuation algebras with functional duals

Wesley Fussner and Alessandra Palmigiano

Dedicated to Ralph Freese, Bill Lampe, and J.B. Nation.

Abstract. We employ the theory of canonical extensions to study residuation algebras whose associated relational structures are *functional*, i.e., for which the ternary relations associated to the expanded operations admit an interpretation as (possibly partial) functions. Providing a partial answer to a question of Gehrke, we demonstrate that functionality is not definable in the language of residuation algebras (or even residuated lattices), in the sense that no equational or quasi-equational condition in the language of residuation algebras is equivalent to the functionality of the associated relational structures. Finally, we show that the class of Boolean residuation algebras such that the atom structures of their canonical extensions are functional generates the variety of Boolean residuation algebras.

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Keywords. Residuation algebras, Canonical extensions, Definability of functionality.

1. Introduction

In the context of a research program aimed at establishing systematic connections between the foundations of automata theory in computer science and duality theory in logic, in [5], Gehrke specializes extended Stone and Priestley dualities in the tradition of [8] so as to capture *topological algebras* as dual

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spaces. A topological algebra of similarity type τ is an algebra of type τ in the category of topological spaces (i.e. it is a topological space endowed with continuous operations for each $f \in \tau$), and Gehrke characterizes topological algebras based on Stone spaces as those relational Stone spaces (see, e.g., [8]) in which the $(n+1)$-ary relations dually corresponding to n-ary operations on Boolean algebras are *functional*. An analogous result is also obtained for topological algebras based on Priestley spaces. In particular, focusing the presentation on residuation algebras (see Definition 2.1), the additional operations on distributive lattices are characterized for which the dual relations are functional (see [5, Proposition 3.16]). These results are formulated and proved without explicit reference to the theory of canonical extensions.

The present contribution is motivated by a question raised in [5, end of Section 3.2], viz. whether the conditions of the statement of [5, Proposition 3.16] are equivalent to a first-order property of residuation algebras. To address this question, we have recast some of the notions and facts pertaining to residuation algebras in the language and theory of canonical extensions, which allows for these facts to be reformulated independently of specific duality-theoretic representations. Our contributions are as follows.

Firstly, we obtain a more modular and transparent understanding of how the validity of the equation $a\backslash(b \vee c) = (a\backslash b) \vee (a\backslash c)$ forces the functionality of the dual relation. Because $(a\backslash b) \vee (a\backslash c) \leq a\backslash(b \vee c)$ holds in every residuation algebra by the monotonicity of $\backslash$ in its second coordinate, $a\backslash(b\vee c) = (a\backslash b)\vee(a\backslash c)$ is equivalent to the inequality $a\backslash(b\vee c) \leq (a\backslash b)\vee(a\backslash c)$. In each setting (Boolean, distributive), the validity of this inequality forces the product of join-irreducible elements (which is a closed element, by the general theory of π-extensions of normal dual operators) to be either $\perp$ or finitely join prime (cf. Proposition 2.7). Moreover, prime closed elements of the canonical extension of a general lattice expansion are completely join-irreducible (see Lemma 2.6). The functionality of the dual relation is obtained as a consequence of these two facts, of which only the first depends on the validity of the inequality above.

Secondly, we provide a partial answer to the initial question. Specifically, functionality cannot be captured by any equational or quasiequational condition, since there is no first-order *universal* sentence in the language of residuation algebras (or even residuated lattices in the sense of [4]) that is equivalent to functionality (see Proposition 2.10). Investigating further along this line, we show that the variety of Boolean residuation algebras is generated by the class of Boolean residuation algebras such that the atom structures of their canonical extensions are functional (see Proposition 2.11).

Thirdly and finally, we articulate a version of [5, Proposition 3.16]—reformulated in a purely algebraic fashion—in which one of the equivalent conditions in the statement is made weaker, and the corresponding part of the proof is simplified and rectified (see Proposition 3.1).

2. Residuation algebras and their canonical extensions

Definition 2.1 (cf. [5], Definition 3.14). A *residuation algebra* is a structure $\mathbf{A} = (A, \backslash, /)$ such that A is a bounded distributive lattice, $\backslash$ and $/$ are binary operations on A such that $\backslash$ (resp. $/$) preserves finite (hence also empty) meets in its second (resp. first) coordinate, and

$$b \leq a\backslash c \quad \text{iff} \quad a \leq c/b.$$

for all $a, b, c \in A$. Note that whenever $(A, \backslash, /)$ is a residuation algebra, we denote the meet, join, least element, and greatest element of the bounded distributive lattice A by $\wedge$, $\vee$, $\bot$, and $\top$, respectively. We assume that $\bot$ and $\top$ are distinguished in the language.

Setting the stage for their application in the sequel, we recall the definition and basic theory of canonical extensions of lattices and their expansions by additional operations.

Definition 2.2 (cf. [7], Definition 1). Given a lattice L, a *canonical extension* of L is a complete lattice L^δ with a lattice embedding $L \hookrightarrow L^\delta$ such that:

(1) L^δ is completely join-generated by the meet-closure $K(L^\delta)$ of L in L^δ, and is completely meet-generated by the join-closure $O(L^\delta)$ of L in L^δ.
(2) If $S, T \subseteq L$ with $\bigwedge S \leq \bigvee T$, then there exist finite subsets $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$.

Property (1) is called *density*, and property (2) is called *compactness*.

Because each lattice L has a canonical extension and canonical extensions are unique up to an isomorphism fixing L (see, e.g., [7, Theorem 1]), we will often call L^δ *the* canonical extension of L. Additionally, if $\mathbf{A}= (A, \backslash, /)$ is a residuation algebra, then $\backslash$ and $/$ may be extended to binary operations $\backslash^\pi$ and $/^\pi$ on A^δ defined, for every $o \in O(A^\delta)$ and $k \in K(A^\delta)$, as follows:

$$k\backslash^\pi o = \bigvee\{a\backslash b \mid a, b \in A, k \leq a \text{ and } b \leq o\},$$
$$o/^\pi k = \bigvee\{a/b \mid a, b \in A, a \leq o \text{ and } k \leq b\},$$

and then for every $u, v \in A^\delta$,

$$u\backslash^\pi v = \bigwedge\{k\backslash o \mid o \in O(A^\delta), k \in K(A^\delta), k \leq u \text{ and } v \leq o\},$$
$$u/^\pi v = \bigwedge\{o/k \mid o \in O(A^\delta), k \in K(A^\delta), k \leq v \text{ and } u \leq o\}.$$

The operation $\backslash^\pi$ (resp. $/^\pi$) is often called the π-*extension* of $\backslash$ (resp. $/$).

For a thorough treatment of canonical extensions of lattices bearing residuated operations, we refer the reader to [4, Chapter 6] and [6]. For the present purposes, we note that if $\mathbf{A} = (A, \backslash, /)$ is a residuation algebra, then $\mathbf{A}^\delta = (A^\delta, \backslash^\pi, /^\pi)$ is a residuation algebra as well. We call the residuation algebra $\mathbf{A}^\delta$ the *canonical extension of* $\mathbf{A}$. Moreover, the residuation condition of Definition 2.1 implies that $\backslash$ (resp. $/$) converts finite (hence empty) joins in its first (resp. second) coordinate into meets. Together with the meet-preservation properties mentioned in Definition 2.1, this implies (see [6, Lemma

4.6]) that $\backslash^\pi$ and $/^\pi$ preserve *arbitrary* meets in their order-preserving coordinates and reverse *arbitrary* joins in their order-reversing coordinates. Since A^δ is a complete lattice, this implies that an operation $\cdot : A^\delta \times A^\delta \to A^\delta$ exists which is completely join-preserving in each coordinate and such that for all $u, v, w \in A^\delta$,

$$v \leq u\backslash^\pi w \quad \text{iff} \quad u \cdot v \leq w \quad \text{iff} \quad u \leq w/^\pi v.$$

Hence, $\mathbf{A}^\delta$ is a complete residuation algebra endowed with the structure of a complete lattice-ordered residuated groupoid. In fact, because $\mathbf{A}$ embeds into $\mathbf{A}^\delta$ as a residuation algebra, it follows that each residuation algebra is isomorphic to a subreduct of a lattice-ordered residuated groupoid. Thus, up to isomorphism residuation algebras are exactly the multiplication-free subreducts of lattice-ordered residuated groupoids. Moreover, $\cdot$ restricts to the elements of the meet-closure $K(A^\delta)$ of A in A^δ (see [1, Lemma 10.3.1]).

To state the following definition, we recall that if L is a lattice, then $x \in L$ is *completely join-irreducible* if $x = \bigvee S$ implies $x \in S$ for any $S \subseteq L$. If L is distributive, L^δ is completely distributive and hence completely join-irreducible elements are *completely join-prime*, i.e. for any $S \subseteq L^\delta$, if $x \leq \bigvee S$ then $x \leq s$ for some $s \in S$.

Definition 2.3. For any residuation algebra $\mathbf{A}$ as above, its associated relational *dual structure* $\mathbf{A}^\delta_+ := (J^\infty(A^\delta), \geq, R)$ is based on the set $J^\infty(A^\delta)$ of the completely join-irreducible elements of A^δ with the converse order inherited from A^δ, and endowed with the ternary relation R on $J^\infty(A^\delta)$ defined for $x, y, z \in J^\infty(A^\delta)$ by

$$R(x, y, z) \quad \text{iff} \quad x \leq y \cdot z.$$

Such an R is *functional* if $y \cdot z \in J^\infty(A^\delta) \cup \{\bot\}$ for all $y, z \in J^\infty(A^\delta)$, in which case we also say that $\mathbf{A}^\delta_+$ is *functional*, and is *functional and defined everywhere* if $y \cdot z \in J^\infty(A^\delta)$ for all $y, z \in J^\infty(A^\delta)$. In this case, we say that $\mathbf{A}^\delta_+$ is *total*.

Remark 2.4. Note that functional relations as defined in [5, Definition 3.1] correspond to relations which are functional and defined everywhere in the present paper.

Group relation algebras, full relation algebras over a given set, and semilinear residuated lattices give examples of residuation algebras whose dual structures are functional.

Notice that by allowing the possibility that $y \cdot z = \bot$, we are allowing the set $R^{-1}[y, z] := \{x \mid R(x, y, z)\}$ to be empty for some $y, z \in J^\infty(A^\delta)$. We emphasize that it is not uncommon that $y \cdot z = \bot$ for $y, z \in J^\infty(A^\delta)$. For instance, in any finite Boolean algebra, where $\backslash$ and $/$ coincide with the Boolean implication and $\cdot$ coincides with $\wedge$, the product of two distinct join-irreducible elements is $\bot$. Examples of algebras in which the product of join-irreducibles may be $\bot$ are also found among MV-algebras and Sugihara monoids. A residuation algebra $\mathbf{A}$ as above *has no zero-divisors* if $x \cdot y \neq \bot$ for all $x, y \in J^\infty(A^\delta)$.

The next two lemmas give a useful connection between the duality-theoretic perspective of [5] and the setting of canonical extensions. Specifically, they capture in a purely algebraic fashion one key property of prime filters of *general* lattices, namely that each prime filter induces a maximal filter/ideal pair, given by itself and its complement. This fact underlies why primeness implies join-irreducibility. Recall that if L is a lattice and $u \in L$, then u is *finitely prime* if $u \neq \bot$ and for all $v, w \in L$, if $u \leq v \vee w$ then $u \leq v$ or $u \leq w$.

Lemma 2.5. *For any lattice L, if $k \in K(L^\delta)$ is finitely prime and $o = \bigvee\{b \in L \mid b \ngeq k\}$, then $k \nleq o$.*

Proof. By way of contradiction, suppose that $\bigwedge\{a \in L \mid k \leq a\} = k \leq o$. Then by compactness, there exist finite sets $A \subseteq \{a \in L \mid k \leq a\}$ and $B \subseteq \{b \in L \mid b \ngeq k\}$ such that

$$a' = \bigwedge A \leq \bigvee B = b'.$$

Then $a' \geq k$, and $b' \ngeq k$ (for if not, then by the primeness of k we would have $b \geq k$ for some $b \in B$, a contradiction). But then $k \leq a' \leq b'$, so $k \leq b'$, a contradiction. This settles the lemma. □

Lemma 2.6. *In any lattice L, if $k \in K(L^\delta)$ is finitely prime, then $k \in J^\infty(L^\delta)$.*

Proof. By denseness it is enough to show that if $k = \bigvee S$ for $S \subseteq K(L^\delta)$, then $k = s$ for some $s \in S$. Let $o = \bigvee\{a \in L \mid a \ngeq k\}$, and, toward a contradiction, assume that $s < k$ for all $s \in S$. The assumption that $S \subseteq K(L^\delta)$ implies that for each $s \in S$,

$$s = \bigwedge\{a \in L \mid a \geq s\},$$

whence for all $s \in S$ there exists $a_s \in L$ such that $a_s \geq s$ and $a_s \ngeq k$. Hence, $a_s \leq o = \bigvee\{a \in L \mid a \ngeq k\}$ for each $s \in S$, and so $\bigvee\{a_s \mid s \in S\} \leq o$. Therefore,

$$o \geq \bigvee\{a_s \mid s \in S\} \geq \bigvee S = k,$$

which contradicts Lemma 2.5, proving the claim. □

While the lemmas above hold for *general* lattices, the next proposition makes use of residuation algebras being based on distributive lattices.

Proposition 2.7. *For any residuation algebra $\mathbf{A}$, if $\mathbf{A} \models a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c)$, then the dual structure $\mathbf{A}_+^\delta$ is functional.*

Proof. The inequality $a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c)$ is Sahlqvist (see [1, Definition 3.5]), and hence canonical (see [1, Theorems 7.1 and 8.8]). That is, the assumption that $\mathbf{A} \models a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c)$ implies that $\mathbf{A}^\delta \models a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c)$. Our aim is to show that for all $x, y \in J^\infty(A^\delta)$, if $x \cdot y \neq \bot$ then $x \cdot y \in J^\infty(A^\delta)$. From $x, y \in J^\infty(A^\delta) \subseteq K(A^\delta)$, it follows that $x \cdot y \in K(A^\delta)$ (see the discussion after Definition 2.1). Hence, by Lemma 2.6 it is enough to show that $x \cdot y$ is finitely prime. Suppose that $x \cdot y \leq \bigvee S$ for a finite subset $S \subseteq A^\delta$. By residuation, $y \leq x\backslash^\pi \bigvee S \leq \bigvee\{x\backslash^\pi s \mid s \in S\}$ (here we are using

$\mathbf{A}^\delta \models a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c))$. By the primeness of y (here we are using distributivity), this implies that $y \leq x\backslash s$ for some $s \in S$, i.e., $x \cdot y \leq s$ for some $s \in S$, which concludes the proof. □

The situation in which the dual relation is functional and defined everywhere is captured by the following corollary, which is an immediate consequence of the proposition above.

Corollary 2.8. *For any residuation algebra* $\mathbf{A}$*, if* $\mathbf{A}$ *has no zero-divisors and* $\mathbf{A} \models a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash c)$*, then* $\mathbf{A}_+^\delta$ *is total (see Definition 2.3).*

Although the inequality $a\backslash(b \vee c) \leq (a\backslash b) \vee (a\backslash b)$ forces the functionality of $\mathbf{A}_+^\delta$, we observe that neither this nor any other equational condition may characterize functionality. Indeed, there is no first-order universal sentence in the language of residuation algebras that is equivalent to functionality, as the following example demonstrates.

Example 2.9. Consider the group $\mathbb{Z}_3$ and its complex algebra, i.e., the algebra $\mathbf{A} = (\mathcal{P}(\mathbb{Z}_3), \cap, \cup, \cdot, \backslash, /, \{0\})$, where for $A, B \in \mathcal{P}(\mathbb{Z})$,

$$\begin{aligned} A \cdot B &= \{a + b \mid a \in A, b \in B\}, \\ A\backslash B &= \{c \mid A \cdot \{c\} \subseteq B\}, \\ A/B &= \{c \mid \{c\} \cdot B \subseteq A\}. \end{aligned}$$

The algebra $\mathbf{A}$ is a finite residuation algebra (indeed, a residuated lattice [4]), hence $\mathbf{A}^\delta = \mathbf{A}$. Moreover, $\{n\}\cdot\{m\} = \{n+m\}$ for all $n, m \in \mathbb{Z}_3$ implies that the ternary relation R on $J^\infty(\mathcal{P}(\mathbb{Z}_3))$ arising from $\cdot$ is functional and defined everywhere, hence $\mathbf{A}_+^\delta$ is functional, and even total. However, $\{\emptyset, \{0\}, \{1, 2\}, \mathbb{Z}_3\}$ is the universe of a subalgebra of $\mathbf{A}$ in both the language of residuated lattices and residuation algebras in which the product of join-irreducible elements may be neither $\perp$ nor join-irreducible since, for example, $\{1, 2\} \cdot \{1, 2\} = \mathbb{Z}_3$ is not join-irreducible.

From the example above, we obtain the following.

Proposition 2.10. *There is no universal first-order property in the language of residuation algebras that characterizes functionality. That is, there is no set* Σ *of universal first-order sentences such that for each residuation algebra* $\mathbf{A}$*,* $\mathbf{A} \models \Sigma$ *if and only if* $\mathbf{A}_+^\delta$ *is functional.*

Proof. If on the contrary Σ were a set a universal first-order sentences characterizing functionality, then $\mathbf{A} \models \Sigma$, where $\mathbf{A}$ is the residuation algebra of Example 2.9. Because the satisfaction of universal first-order sentences is inherited by subalgebras, this implies that the subalgebra $\mathbf{B}$ of $\mathbf{A}$ with universe $\{\emptyset, \{0\}, \{1, 2\}, \mathbb{Z}_3\}$ also satisfies Σ. But $\mathbf{B}_+^\delta$ is not functional, as discussed in Example 2.9. This is a contradiction, and the result follows. □

Two remarks are in order. First, Example 2.9 actually provides more than the statement of Proposition 2.10 makes explicit: Because the algebras of Example 2.9 are residuated lattice-ordered groupoids with multiplicative neutral element, the result articulated above applies even for the expanded language of these structures. Second, Example 2.9 also shows by duality that the class of functional duals of residuation algebras is not closed under taking p-morphic images. An application of the characterization of modal definability offered in [3, Theorem 50] therefore provides another proof that functionality is not characterized by an equational condition in the language of residuation algebras.

In light of the fact that residuation algebras with functional duals do not form a variety, it is natural to ask which subvariety of residuation algebras they generate. We conclude this section with a result in this direction. Define a *Boolean residuation algebra* to be an expansion of a residuation algebra $(A, \backslash, /)$ by a unary operation $'$ such that $x \wedge x' = \bot$ and $x \vee x' = \top$, i.e., such that A is a Boolean algebra with complementation operation $'$.

Proposition 2.11. *Let $\mathcal{C}$ be the class of Boolean residuation algebras $\mathbf{A}$ such that $\mathbf{A}_+^\delta$ is functional. Then the variety of Boolean residuation algebras is generated by $\mathcal{C}$.*

Proof. It suffices to show that every Boolean residuation algebra embeds in some member of $\mathcal{C}$, so let $\mathbf{A}$ be a Boolean residuation algebra. Then $\mathbf{A}$ embeds in $\mathbf{A}^\delta$. Since Boolean algebras are closed under taking canonical extensions, we have that $\mathbf{A}^\delta$ is again a Boolean residuation algebra, and we consider it as a Boolean lattice-ordered residuated groupoid. [9, Theorem 3.20] provides that every Boolean lattice-ordered residuated groupoid embeds in the complex algebra $\mathbf{B}$ of a partial groupoid. This implies that $\mathbf{A}$ embeds in $\mathbf{B}$, and because $\mathbf{B} \in \mathcal{C}$ the result follows. $\square$

3. Characterizing functionality

The following proposition emends [5, Proposition 3.16]. Items (2) and (3) amount to equivalent reformulations of the corresponding items in the setting of canonical extensions. Item (1) is weaker than the corresponding item in [5, Proposition 3.16], and does not stipulate that the operation $\cdot$ gives rise to a functional relation defined everywhere (see Definition 2.3). The proof of (1)⇒(2) is essentially the same as the corresponding proof in [5, Proposition 3.16]; we observe that it goes through also under this relaxed assumption. The proof of (3)⇒(1) is simpler than the corresponding proof in [5, Proposition 3.16], and is where the emendation takes place.

Proposition 3.1. *The following conditions are equivalent for any residuation algebra $\mathbf{A} = (A, \backslash, /)$:*

(1) *The relational structure $\mathbf{A}_+^\delta$ is functional (see Definition 2.3).*
(2) $\forall a, b, c \in A, \forall x \in J^\infty(A^\delta)[x \leq a \Rightarrow \exists a'[a' \in A \;\&\; x \leq a' \;\&\; a\backslash(b \vee c) \leq (a'\backslash b) \vee (a'\backslash c)]$.
(3) *For all $x \in J^\infty(A^\delta)$, the map $x\backslash^\pi(\_) : O(A^\delta) \to O(A^\delta)$ is $\vee$-preserving.*

Proof. (1)⇒(2): Let $a, b, c \in A$, and $x \in J^\infty(A^\delta)$ such that $x \leq a$. We need to find some $a' \in A$ such that $x \leq a'$ and $a\backslash(b \vee c) \leq (a'\backslash b) \vee (a'\backslash c)$. If $y \in J^\infty(A^\delta)$

and $y \le a\backslash(b \vee c)$ i.e. $a \cdot y \le b \vee c$, then $x \cdot y \le b \vee c$. By assumption (1) and because in distributive lattices $x, y \in J^\infty(A^\delta)$ are prime, this implies that $x \cdot y \le b$ or $x \cdot y \le c$, both in the case in which $x \cdot y = \bot$ and in case $x \cdot y \ne \bot$. This can be equivalently rewritten as

$$y \le x\backslash^\pi b = \bigvee\{a\backslash b \mid a \in A \text{ and } x \le a\}$$
$$\text{or } y \le x\backslash^\pi c = \bigvee\{a\backslash c \mid a \in A \text{ and } x \le a\}$$

(notice that here we are applying the simpler definition of $\backslash^\pi$ restricted to $x \in J^\infty(A^\delta) \subseteq K(A^\delta)$ and $b, c \in A \subseteq O(A^\delta)$). Since $y \in J^\infty(A)$, this implies that $y \le a_y\backslash b$ or $y \le a_y\backslash c$ for some $a_y \in A$ such that $x \le a_y$, which implies that $y \le (a_y\backslash b) \vee (a_y\backslash c)$. Hence, given that $a_y \in A$ and $x \le a_y$ for all such a_y,

$$\begin{aligned} a\backslash(b \vee c) &= \bigvee\{y \in J^\infty(A) \mid y \le a\backslash(b \vee c)\} \\ &\le \bigvee\{(a\backslash b) \vee (a\backslash c) \mid a \in A \text{ and } x \le a\}. \end{aligned}$$

Hence, by compactness, and the antitonicity of $\backslash$ in the first coordinate,

$$a\backslash(b \vee c) \le \bigvee\{(a_i\backslash b) \vee (a_i\backslash c) \mid 1 \le i \le n\} \le (a'\backslash b) \vee (a'\backslash c)$$

where $a' := \bigwedge_{i=1}^n a_i \in A$ and $x \le a'$, as required.

(2)⇒(3): Let $x \in J^\infty(A^\delta)$ and $o_1, o_2 \in O(A^\delta)$. We need to prove that

$$x\backslash^\pi(o_1 \vee o_2) \le (x\backslash^\pi o_2) \vee (x\backslash^\pi o_2). \tag{3.1}$$

By definition of $\backslash^\pi$,

$$\begin{aligned} x\backslash^\pi(o_1 \vee o_2) &= \bigvee\{a\backslash d \mid a, d \in A \text{ and } x \le a \text{ and } d \le o_1 \vee o_2\}, \\ x\backslash^\pi o_1 &= \bigvee\{a'\backslash b \mid a', b \in A \text{ and } x \le a' \text{ and } b \le o_1\}, \\ x\backslash^\pi o_2 &= \bigvee\{a'\backslash c \mid a', c \in A \text{ and } x \le a' \text{ and } c \le o_2\}. \end{aligned}$$

Thus, to prove (3.1) it is enough to show that, for all $a, d \in A$ such that $x \le a$ and $d \le o_1 \vee o_2$, some $a', b, c \in A$ exist such that $x \le a'$, $b \le o_1$, $c \le o_2$ and $a\backslash d \le (a'\backslash b) \vee (a'\backslash c)$. From $d \le o_1 \vee o_2 = \bigvee\{b \in A \mid b \le o_1\} \vee \bigvee\{c \in A \mid c \le o_2\}$ we get by compactness that $d \le b \vee c$ for some $b, c \in A$ such that $b \le o_1$ and $c \le o_2$. Then, by assumption (2), $a\backslash d \le a\backslash(b \vee c) \le (a'\backslash b) \vee (a'\backslash c)$ for some $a' \in A$ such that $x \le a'$, as required.

(3)⇒(1): Let $x, y \in J^\infty(A^\delta)$. Then $x \cdot y \in K(A^\delta)$ because of general facts about canonical extensions of maps. Hence, by Lemma 2.6, it is enough to show that, for all $u, v \in A^\delta$, if $x \cdot y \ne \bot$ and $x \cdot y \le u \vee v$ then $x \cdot y \le u$ or $x \cdot y \le v$. By denseness, it is enough to prove the claim for $u, v \in O(A^\delta)$, and by compactness, it is enough to prove the claim for $u = b \in A$ and $v = c \in A$. The assumption $x \cdot y \le b \vee c$ can be equivalently rewritten as $y \le x\backslash^\pi(b \vee c) = (x\backslash^\pi b) \vee (x\backslash^\pi c)$, the equality due to assumption (3). The primeness of y yields $y \le x\backslash^\pi b$ or $y \le x\backslash^\pi c$, i.e. $x \cdot y \le b$ or $x \cdot y \le c$, as required. □

4. Conclusion

The class of residuation algebras with functional duals is not a universal class (much less a variety) according to Proposition 2.10, but it remains open whether the property of having a functional dual may be expressed by a first-order condition in the language of residuation algebras. We pose three other questions that are implicated by the foregoing analysis. First, what is the variety generated by the class of residuation algebras with functional duals, and (in particular) do the residuation algebras with functional duals generate the variety of all residuation algebras? We have a partial positive answer relative to the class of Boolean residuation algebras. Second, can the treatment given in this paper be extended to residuated algebraic structures with non-distributive lattice reducts? Third, given that the canonicity of Sahlqvist inequalities is key to this result, and given that the core inequality expresses the additivity of a right residual map in its order-preserving coordinates, can we extend this result to signatures of additive or multiplicative connectives on the basis of the (constructive) canonicity theory for normal and regular connectives developed in [2]? We do not presently know the answer to these questions, but their resolution would deepen our understanding of functionality and promise interesting applications.

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